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Survey Article

Greek Geometry, Rational Trigonometry, and the Snellius – Pothenot Surveying Problem

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Abstract: Pythagoras' theorem, Euclid's formula for the area of a triangle as one half the base times the height, and Heron's or Archimedes' formula are amongst the most important and useful results of ancient Greek geometry. Here we look at all three in a new and improved light, replacing *distance* with *quadrance*, and *angle* with *spread*. As an application of this simpler and more elegant *rational trigonometry*, we show how the famous surveying problem of Snellius and Pothenot, also called resection, can be simplified by a purely algebraic approach.

Keywords: Rational trigonometry, Pythagoras' theorem, Heron's formula, quadrance, spread, resection, Snellius Pothenot problem

2010 Mathematics Subject Classification: 51Fxx, 51Nxx

Three ancient Greek theorems

There are three classical theorems about triangles that every student meets, or at least ought to meet. Here are the usual formulations, in terms of a triangle $\overline{A_1A_2A_3}$ with side lengths $d_1 \equiv |A_2, A_3|$, $d_2 \equiv |A_1, A_3|$ and $d_3 \equiv |A_1, A_2|$.

Pythagoras' theorem *The triangle* $\overline{A_1A_2A_3}$ *has a right angle at* A_3 *precisely when*

$$d_1^2 + d_2^2 = d_3^3.$$

Euclid's theorem *The area of a triangle is one half the base times the height.*

Heron's theorem *If* $s \equiv (d_1 + d_2 + d_3)/2$ *is the semi-perimeter of the triangle, then*

$$\text{area} = \sqrt{s(s-d_1)(s-d_2)(s-d_3)}.$$

In this paper we will recast all three in simpler and more general forms by removing unnecessary irrationalities. As a reward, we find that *rational trigonometry* falls into our laps, essentially for free. Our reformulation extends to a general field (not of characteristic two), to higher dimensions, and even with an arbitrary quadratic form, as in [3] and [4]. We then apply these ideas to give a completely algebraic solution to probably the most famous classical problem in surveying: the *resection problem of Snellius and Pothenot* (see for example [1]).

Pythagoras' theorem

Euclid and other ancient Greeks rightly regarded *area*, not *distance*, as the fundamental quantity in planar geometry. They worked with a straightedge and compass in their constructions, not a ruler, and a line segment was measured by the area of a square on it. Two line segments were considered equal if they were congruent, but this was independent of a direct notion of distance measurement.

Pythagoras' theorem, as it appears in Propositions 47 and 48 of Book 1 of Euclid's *Elements*, states that the area of the square on one side of a triangle is the sum of the areas of the squares on the other two sides precisely when the triangle is a right triangle. This formulation in terms of areas as opposed to lengths is often neglected today, but with a sheet of graph paper it is still an attractive way to introduce students to the subject, as the area of many simple figures can be computed by subdividing and translating, while lengths are generally impossible to compute correctly.

The squares on the sides of triangle $\overline{A_1A_2A_3}$ shown in Figure 1 have areas $5, 20$ and 25. The largest square, for example, can be seen as four triangles which can be rearranged to get two rectangles each of sides 3×4, together with a 1×1 square, for a total area of 25. So Pythagoras' theorem can be established by *counting*, and

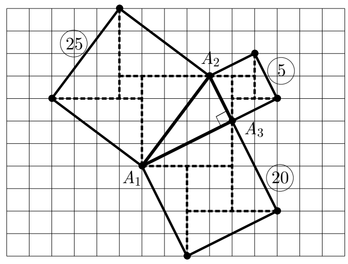

Figure 1: Pythagoras' theorem using areas

the use of irrational numbers to describe lengths is unnecessary for a triangle with rational coordinates.

Following the Greek terminology of 'quadrature', we will use the word *quadrance* and the symbol Q to denote the area of a square constructed on a line segment. In coordinates, if $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$, then the **quadrance** between A_1 and A_2 is

$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 .$$

So for example the quadrance between the points $[0, 0]$ and $[1, 2]$ is $Q = 5$. The usual distance is the square root of the quadrance, and requires a prior theory of irrational numbers. Clearly the irrational number $\sqrt{5} \approx 2.236\,067\,977\ldots$ is a far more sophisticated and complicated object than the natural number 5.

In statistics, variance is more natural than standard deviation. In quantum mechanics, wavefunctions are more basic than probability amplitudes. In harmonic analysis, L^2 is more pleasant than L^1. In geometry, *quadrance is more fundamental than distance.*

For a triangle $\overline{A_1A_2A_3}$ we define the quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$. Here then is *Pythagoras' theorem as the Greeks viewed it*:

Theorem 1 (Pythagoras). *The lines A_1A_3 and A_2A_3 of the triangle $\overline{A_1A_2A_3}$ are perpendicular precisely when*

$$Q_1 + Q_2 = Q_3.$$

With this formulation, the theorem extends to arbitrary fields, higher dimensions, and to more general quadratic forms, such as the Minkowski form of special relativity (see [6]). Furthermore, a relatively simple deformation gives Pythagoras' theorem in both hyperbolic and elliptic geometries (see [7], [8]).

Euclid's theorem

Let's now see how we might remove some of the irrationalities inherent in Euclid's theorem. The area of the triangle $\overline{A_1A_2A_3}$ in Figure 2 is one half of the area of

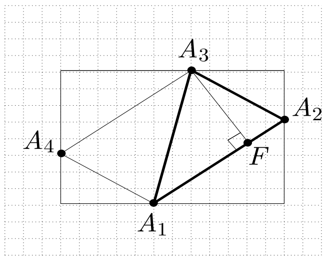

Figure 2: A triangle and an associated parallelogram

the associated parallelogram $\overline{A_1A_2A_3A_4}$. The latter area may be calculated by removing from the outer 12×8 rectangle four triangles, which can be combined to form two rectangles, one 5×3 and the other 7×5. The area of $\overline{A_1A_2A_3}$ is thus $(96 - 15 - 35)/2 = 23$.

To apply the one-half base times height rule, the base $\overline{A_1A_2}$ by Pythagoras has length

$$d_3 = |A_1, A_2| = \sqrt{7^2 + 5^2} = \sqrt{74} \approx 8.602\,325\,267\,04\ldots.$$

To find the length h of the altitude $\overline{A_3F}$, set the origin to be at A_1, then the line A_1A_2 has Cartesian equation $5x - 7y = 0$ while $A_3 = [2, 8]$. A well-known result from coordinate geometry then states that the distance $h = |A_3, F|$ from A_3 to the line A_1A_2 is

$$h = \frac{|5 \times 2 - 7 \times 8|}{\sqrt{5^2 + 7^2}} = \frac{46}{\sqrt{74}} \approx 5.347\,391\,382\,22\ldots.$$

If an engineer doing this calculation works with the surd forms of both expressions, she will notice that the two occurrences of $\sqrt{74}$ conveniently cancel when she takes one half the product of d_3 and h, giving an area of 23. However if she works immediately with the decimal forms, she may be surprised that her calculator gives

$$\text{area} \approx \frac{8.602\,325\,267\,04\ldots \times 5.347\,391\,382\,22\ldots}{2} \approx 23.000\,000\,000\,01.$$

The usual formula forces us to descend to the level of irrational numbers and square roots, even when the eventual answer is a natural number, and this introduces unnecessary approximations and inaccuracies into the subject. It is not hard to see how the use of quadrance allows us to reformulate the result.

Theorem 2 (Euclid). *The square of the area of a triangle is one-quarter the quadrance of the base times the quadrance of the corresponding altitude.*

As a formula, this would be

$$(\text{area})^2 = \frac{Q \times H}{4}$$

where Q is the quadrance of the base and H is the quadrance of the altitude to that base.

Heron's or Archimedes' Theorem

The same triangle $\overline{A_1A_2A_3}$ of the previous section has side lengths

$$d_1 = \sqrt{34} \qquad d_2 = \sqrt{68} \qquad d_3 = \sqrt{74}.$$

The semi-perimeter s, defined to be one half of the sum of the side lengths, is then

$$s = \frac{\sqrt{34} + \sqrt{68} + \sqrt{74}}{2} \approx 11.339\,744\,206\,6\ldots.$$

Using the usual Heron's formula, a computation with the calculator shows that

$$\text{area} = \sqrt{s\left(s - \sqrt{34}\right)\left(s - \sqrt{68}\right)\left(s - \sqrt{74}\right)} \approx 23.000\,000\,000\,4.$$

Again we have a formula involving square roots in which there appears to be a surprising integral outcome. Let's now give a new and improved form of Heron's theorem, with also a more appropriate name. Arab sources suggest that Archimedes knew Heron's formula earlier, and the greatest mathematician of all time deserves credit for more than he currently gets.

Theorem 3 (Archimedes). *The area of a triangle* $\overline{A_1A_2A_3}$ *with quadrances* Q_1, Q_2 *and* Q_3 *is determined by the formula*

$$16 \times (\text{area})^2 = (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

In our example the triangle has quadrances $34, 68$ and 74, each obtained by Pythagoras' theorem. So Archimedes' theorem states that

$$16 \times (\text{area})^2 = (34 + 68 + 74)^2 - 2\left(34^2 + 68^2 + 74^2\right) = 8464$$

and this gives an area of 23. In rational trigonometry, the quantity

$$\mathcal{A} \equiv (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$$

is called the **quadrea** of the triangle, and turns out to be the single most important number associated to a triangle. Note that

$$\mathcal{A} = 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2 = \begin{vmatrix} 2Q_1 & Q_1 + Q_2 - Q_3 \\ Q_1 + Q_2 - Q_3 & 2Q_2 \end{vmatrix}$$
$$= -\begin{vmatrix} 0 & Q_1 & Q_2 & 1 \\ Q_1 & 0 & Q_3 & 1 \\ Q_2 & Q_3 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

It is instructive to see how to go from Heron's formula to Archimedes' theorem. In terms of the side lengths d_1, d_2 and d_3 :

$$\begin{aligned} 16 \times (\text{area})^2 &= (d_1 + d_2 + d_3)(-d_1 + d_2 + d_3)(d_1 - d_2 + d_3)(d_1 + d_2 - d_3) \\ &= \left((d_1 + d_2)^2 - d_3^2\right)\left(d_3^2 - (d_1 - d_2)^2\right) \\ &= \left((d_1 + d_2)^2 + (d_1 - d_2)^2\right)Q_3 - (d_1 + d_2)^2(d_1 - d_2)^2 - Q_3^2 \\ &= 2(Q_1 + Q_2)Q_3 - \left(d_1^2 - d_2^2\right)^2 - Q_3^2 \\ &= 2(Q_1 + Q_2)Q_3 - (Q_1 - Q_2)^2 - Q_3^2 \\ &= (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right). \end{aligned}$$

Archimedes' theorem implies another theorem of very considerable importance.

Theorem 4 (Triple quad formula). *The three points* A_1, A_2 *and* A_3 *are collinear precisely when*

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

The proof is immediate, as collinearity is equivalent to the area of the triangle being zero. The function

$$A(Q_1, Q_2, Q_3) = (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$$

will be called **Archimedes' function**.

Spread between lines

A numerical angle (as opposed to the geometrical configuration made by two intersecting lines), is the ratio of a *circular distance to a linear distance*, and this is a complicated concept, as Euclid well realized. This is why angle measurements play essentially no role in the *Elements*, apart from right angles.

To define an angle properly *you need calculus*, an important point essentially understood by Archimedes. Vagueness about angles, and the accompanying ambiguities in the definitions of the circular functions $\cos\theta$, $\sin\theta$ and $\tan\theta$, weaken the logical coherence of most modern calculus texts.

So there is a reason why classical trigonometry is almost always painful to most students—the subject is *based on the wrong notions*! As a result, mathematics teachers continually rely on $90-45-45$ and $90-60-30$ triangles for examples and test questions, which makes the subject very narrow and repetitive.

Rational trigonometry, developed in [3], see also [6], shows how to enrich and simplify the subject at the same time, leading to greater accuracy and quicker computations. In what follows, we show how the basic ideas follow naturally from our presentation of the theorems of Pythagoras, Euclid and Archimedes.

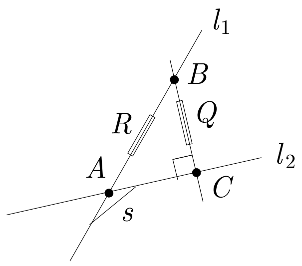

Figure 3: Spread s between two lines l_1 and l_2

The separation between lines l_1 and l_2 is captured by the concept of *spread*, which may be defined as the ratio of two quadrances as follows. Suppose l_1 and l_2 intersect at the point A, as in Figure 3. Choose a point $B \neq A$ on one of the lines, say l_1, and let C be the foot of the perpendicular from B to l_2. Then the **spread** s between l_1 and l_2 is

$$s = s(l_1, l_2) \equiv \frac{Q(B,C)}{Q(A,B)} = \frac{Q}{R}.$$

This ratio is independent of the choice of B, by a theorem that goes back to Thales, and is defined between lines, not rays.

The *spread protractor* in Figure 4 was created by M. Ossmann and is available online at [2].

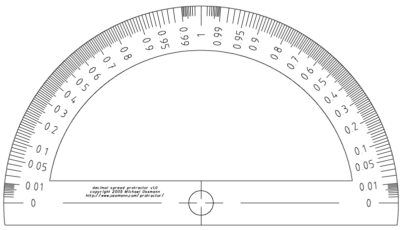

Figure 4: A spread protractor

When lines are expressed in Cartesian form, the spread becomes a *rational expression* in the coefficients of the lines. If the lines l_1 and l_2 have direction vectors $v_1 \equiv (a_1, b_1)$ and $v_2 \equiv (a_2, b_2)$ then a linear algebra calculation (highly recommended!) shows that

$$s(l_1, l_2) = \frac{(a_1 b_2 - a_2 b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

Parallel lines are defined to have spread $s = 0$, while perpendicular lines have spread $s = 1$. You may check that the spread corresponding to 30° or 150° is $s = 1/4$, while the spread corresponding to 60° or 120° is $3/4$. In the triangle $\overline{ABC}$ above, the spread at the vertex A and the spread at the vertex B sum to 1, on account of Pythagoras' theorem.

Rational trigonometry

Let's see how to combine the three ancient Greek theorems above to derive the main laws of trigonometry in this rational form, without any need for transcendental functions. Our general notation will be: a triangle $\overline{A_1A_2A_3}$ has quadrances Q_1, Q_2 and Q_3, as well as corresponding spreads s_1, s_2 and s_3, labelled as in Figure 5.

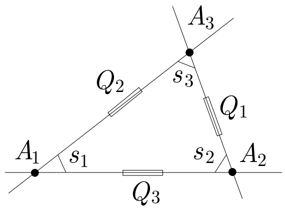

Figure 5: Quadrances and spreads of a triangle

If H_3 is the quadrance of the altitude from A_3 to the line A_1A_2, then Euclid's theorem and the definition of spread give

$$(\text{area})^2 = \frac{Q_3 \times H_3}{4} = \frac{Q_3Q_2s_1}{4} = \frac{Q_3Q_1s_2}{4}.$$

By symmetry, we deduce the following analog of the Sine law.

Theorem 5 (Spread law). *For a triangle with quadrances Q_1, Q_2 and Q_3, and spreads s_1, s_2 and s_3,*

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{4 \times (\text{area})^2}{Q_1Q_2Q_3}.$$

By equating the formulas for $16 \times (\text{area})^2$ given by Euclid's and Archimedes' theorems, we get

$$\begin{aligned} 4Q_1Q_2s_3 &= (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) \\ &= 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2. \end{aligned}$$

This gives the following analog of the Cosine law.

Theorem 6 (Cross law). *For a triangle with quadrances Q_1, Q_2 and Q_3, and spreads s_1, s_2 and s_3,*

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$

Now set

$$D \equiv \frac{Q_1 Q_2 Q_3}{4 \times (\text{area})^2}$$

and substitute $Q_1 = s_1 D$, $Q_2 = s_2 D$ and $Q_3 = s_3 D$ from the Spread law into the Cross law, and cancel the common factor of D^2. The result is the relation

$$(s_1 + s_2 - s_3)^2 = 4 s_1 s_2 (1 - s_3)$$

between the three spreads of a triangle, which can be rewritten more symmetrically as follows.

Theorem 7 (Triple spread formula). *The three spreads s_1, s_2 and s_3 of a triangle satisfy*

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4 s_1 s_2 s_3.$$

This formula is a deformation of the Triple quad formula by a single cubic term, and is the analog in rational trigonometry to the classical fact that the three angles of a triangle sum to (approximately) $3.141\,592\,653\,59\ldots$.

The *Triple quad formula*, *Pythagoras' theorem*, the *Spread law*, the *Cross law* and the *Triple spread formula* are the five main laws of rational trigonometry. We have just seen that these are closely connected with the geometrical work of the ancient Greeks, and really only elementary high school algebra was needed in addition.

As demonstrated at some length in [3], these formulas and a few additional secondary ones suffice to solve the majority of trigonometric problems, usually more simply, more accurately and more elegantly than the classical theory involving $\sin\theta$, $\cos\theta$, $\tan\theta$ and their inverse functions. As shown in [4], [5] and [6], the same formulas extend to geometry over general fields and with arbitrary quadratic forms, and as shown recently in [7] and [8], the main laws of rational trigonometry also in hyperbolic and elliptic geometry are closely related. Let's show how to use rational trigonometry to solve the most famous and important surveying problem.

Snellius-Pothenot problem

The problem of *resection* was originally stated and solved by Snellius (1617) and then by Pothenot (1692). Despite being an interesting application of trigonometry, many mathematics students outside of surveying never see it.

Problem 8 (Snellius/Pothenot, or resection). *The quadrances* Q_1, Q_2 *and* Q_3 *of the known triangle* $\overline{A_1A_2A_3}$ *are known. The spreads* $r_1 \equiv s(BA_2, BA_3)$, $r_2 \equiv s(BA_1, BA_3)$ *and* $r_3 \equiv s(BA_1, BA_2)$ *are measured from point* B. *One wants to find the position of* B *relative to the triangle* $\overline{A_1A_2A_3}$, *in other words to find* $P_1 \equiv Q(B, A_1)$, $P_2 \equiv Q(B, A_2)$ *and* $P_3 \equiv Q(B, A_3)$ *(see Figure 6).*

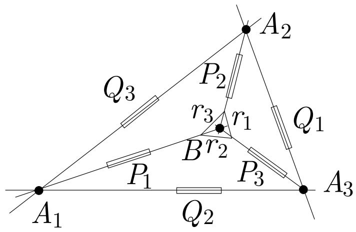

Figure 6: Q_1, Q_2, Q_3 and r_1, r_2, r_3 are known. What are P_1, P_2 and P_3?

The solution presented here has some common features with traditional ones using classical trigonometry, especially the use of Collins' point and a particular circle, but algebraically it is quite different, demonstrating techniques of rational trigonometry. Let's see how to determine P_1 and P_2. Take the circumcircle c_3 of $\overline{A_1A_2B}$ and let H, called **Collin's point**, be the other intersection of c_3 with A_3B, as shown in Figure 7.

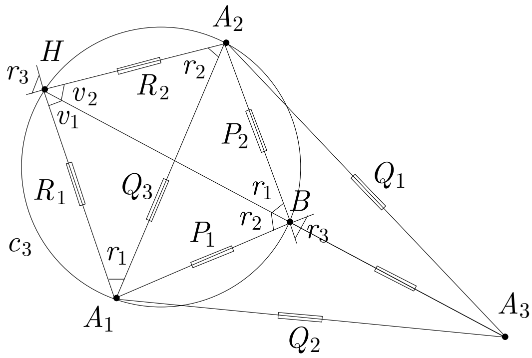

Figure 7: Solution to the resection problem of Snellius and Pothenot

Define the quadrances $R_1 \equiv Q(H, A_1)$, $R_2 \equiv Q(H, A_2)$ and $R_3 \equiv Q(H, A_3)$.

It is then a fact that any two spreads subtended by a chord of a circle are equal (as opposed to the situation with angles, where one needs to worry about the relative positions of the points and the chord). It follows that the spreads $s(A_1H, A_1A_2)$, $s(A_2H, A_2A_1)$ and $s(HA_1, HA_2)$ are respectively r_1, r_2 and r_3. Let $v_1 \equiv s(HA_1, HA_3)$ and $v_2 \equiv s(HA_2, HA_3)$. The Spread law in $\overline{A_1A_2H}$ allows us to write

$$R_1 = r_2Q_3/r_3 \qquad \text{and} \qquad R_2 = r_1Q_3/r_3. \tag{1}$$

The **four point relation**, which goes back to Tartaglia and was also found by Euler, describes the relation between six quadrances formed by a triangle such as $\overline{A_1A_2A_3}$ and an additional point such as H. It is

$$\begin{vmatrix} 2R_1 & R_1+R_2-Q_3 & R_1+R_3-Q_2 \\ R_1+R_2-Q_3 & 2R_2 & R_2+R_3-Q_1 \\ R_1+R_3-Q_2 & R_2+R_3-Q_1 & 2R_3 \end{vmatrix} = 0. \tag{2}$$

In fact the left hand side is the square of the volume of the tetrahedron with edge quadrances Q_1, Q_2, Q_3 and R_1, R_2 and R_3, divided by 288. When working with volumes of polytopes, formulas often involve quadrances.

One can rewrite (2) as the following quadratic equation in R_3 :

$$\left(R_3 - R_1 - R_2 + Q_3 - Q_1 - Q_2 + \frac{(Q_1-Q_2)(R_2-R_1)}{Q_3}\right)^2$$
$$= \frac{A(Q_1,Q_2,Q_3)\,A(R_1,R_2,Q_3)}{4Q_3^2}$$

where A is Archimedes' function. After substituting for the values of R_1 and R_2 from (1) and simplifying, you get the equation

$$(R_3 - C)^2 = D$$

where

$$C = \frac{(Q_1+Q_2+Q_3)(r_1+r_2+r_3) - 2(Q_1r_1 + Q_2r_2 + Q_3r_3)}{2r_3}$$

and

$$D = \frac{r_1r_2\ A(Q_1,Q_2,Q_3)}{r_3}.$$

There are two possible solutions R_3, and for each the Cross law in $\overline{A_1A_3H}$ gives

$$v_1 = 1 - \frac{(R_1+R_3-Q_2)^2}{4R_1R_3}$$

while the Cross law in $\overline{A_2A_3H}$ gives

$$v_2 = 1 - \frac{(R_2 + R_3 - Q_1)^2}{4R_2R_3}.$$

Then the Spread laws in $\overline{A_1BH}$ and $\overline{A_2BH}$ give the required values

$$P_1 = \frac{v_1R_1}{r_2} = \frac{v_1Q_3}{r_3}$$
$$P_2 = \frac{v_2R_2}{r_1} = \frac{v_2Q_3}{r_3}.$$

In [3], from which this derivation is taken, there is an example of the calculations worked out in a particular case, and also the related Hansen's problem is discussed, along with other more elementary surveying problems.

Conclusion

In retrospect, the blind spot first occurred with the Pythagoreans, who initially believed that all of nature should be expressible in terms of natural numbers and their proportions. When they discovered that the ratio of the length of a diagonal to the length of a side of a square was the incommensurable proportion $\sqrt{2} : 1$, they panicked, and according to legend threw the exposer of the secret overboard while at sea.

Had they maintained their beliefs in the workings of the Divine Mind, and concluded that the *squares of the lengths are the crucial quantities in geometry*, then mathematics might have had a significantly different history, the geometry underlying Einstein's special theory of relativity might have been discovered earlier, algebraic geometry would have quite another aspect, and students would be studying a much simpler and more elegant trigonometry.

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