Some Inclusion Relationships of Certain Subclasses of $p$-valent Mermorphic Functions Associated with Certain Integral Operator

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Abstract: In this paper we investigate a family of integral operators defined on the space of $p$-valent meromorphic functions. By making use of these novel integral operators, we introduce and investigate several new subclasses of $p$-valent meromorphic functions. Also we establish some inclusion relationships associated with the aforementioned integral operators. Several interesting integral preserving properties also considered.

Keywords: Analytic functions, $p$-valent meromorphic starlike functions, $p$-valent meromorphic convex functions, $p$-valent meromorphic close-to-convex functions, $p$-valent meromorphic quasi-convex functions, integral operator, Hadamard product, subordination

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1 Introduction

Let $\mathcal{M}_p$ denote the class of $p$-valent functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots \}),$$  \hspace{1cm} (1.1)
which are analytic and meromorphic in the punctured disc

\[ U^* = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} = U \setminus \{0\} . \]

If \( f \) and \( g \) are analytic in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \) or \( f(z) \prec g(z) \), if there exists a Schwarz function \( \omega \), analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) \( (z \in U) \), such that \( f(z) = g(\omega(z)) \) \( (z \in U) \). In particular, if the function \( g \) is univalent in \( U \), \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

For \( 0 \leq \alpha, \beta < p \), we denote by \( \mathcal{MS}_p(\alpha) \), \( \mathcal{MC}_p(\alpha) \), \( \mathcal{MK}_p(\alpha, \beta) \) and \( \mathcal{MQ}_p(\alpha, \beta) \) the subclasses of \( \mathcal{M}_p \) consisting of all analytic functions which are, respectively, \( p \)-valent meromorphic starlike of order \( \alpha \), \( p \)-valent meromorphic convex of order \( \alpha \), \( p \)-valent meromorphic close-to-convex of order \( \alpha \), and type \( \beta \) and \( p \)-valent meromorphic quasi-convex of order \( \alpha \), and type \( \beta \) in \( U \) (see [1] and [4]).

Let \( P \) be the class of all functions \( \phi \) which are analytic and univalent in \( U \) and for which \( \phi(U) \) is convex with \( \phi(0) = 1 \) and \( \Re \{ \phi(z) \} > 0 \) for \( z \in U \).

Making use of the principle of subordination between analytic functions, we introduce the subclasses \( \mathcal{MS}_p(\alpha; \phi) \), \( \mathcal{MC}_p(\alpha; \phi) \), \( \mathcal{MK}_p(\alpha, \beta; \phi, \psi) \) and \( \mathcal{MQ}_p(\alpha, \beta; \phi, \psi) \) of the class \( \mathcal{M}_p \) for \( 0 \leq \alpha, \beta < p \), and \( \phi, \psi \in P \), which are defined by

\[
\mathcal{MS}_p(\alpha; \phi) = \left\{ f \in \mathcal{M}_p : \frac{1}{p - \alpha} \left( \frac{zf'(z)}{f(z)} + \alpha \right) \prec \phi(z) \text{ in } U \right\}, \quad (1.2)
\]

\[
\mathcal{MC}_p(\alpha; \phi) = \left\{ f \in \mathcal{M}_p : \frac{1}{p - \alpha} \left( \frac{zf'(z)'}{f'(z)} + \alpha \right) \prec \phi(z) \text{ in } U \right\}, \quad (1.3)
\]

\[
\mathcal{MK}_p(\alpha, \beta; \phi, \psi) = \left\{ f \in \mathcal{M}_p : \exists g \in \mathcal{MS}_p(\alpha; \phi), \frac{1}{p - \beta} \left( \frac{zf'(z)}{g(z)} + \beta \right) \prec \psi(z) \text{ in } U \right\}, \quad (1.4)
\]

\[
\mathcal{MQ}_p(\alpha, \beta; \phi, \psi) = \left\{ f \in \mathcal{M}_p : \exists g \in \mathcal{MC}_p(\alpha; \phi), \frac{1}{p - \beta} \left( \frac{(zf'(z))'}{g'(z)} + \beta \right) \prec \psi(z) \text{ in } U \right\}. \quad (1.5)
\]

Also let the Hadamard product (or convolution) \( f \ast g \) of two analytic functions \( f(z) \), is defined by (1.1), and \( g(z) \) is defined by

\[
g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in \mathbb{N}), \quad (1.6)
\]
be given (as usual) by
\[(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z). \quad (1.7)\]

For complex parameters \(a_1, \ldots, a_q; b_1, \ldots, b_s\) \((b_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\} ; j = 1, \ldots, s)\), the generalized hypergeometric function \( {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) is given by (see [6])
\[{}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!} \quad (1.8)\]

where \((x)_k\) is the Pochhammer symbol defined (in terms of the Gamma function) by
\[(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)} = \begin{cases} 1 & (k = 0) \\ x(x + 1) \cdots (x + k - 1) & (k \in \mathbb{N}). \end{cases} \quad (1.9)\]

Corresponding to a function \(F_{\mu,p}(a_1, \ldots, a_q; b_1, \ldots, b_s; z)\), defined by
\[F_{\mu,p}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \frac{1 - \mu}{z^p} {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) + \frac{\mu}{z^p} (z^{p+1} {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z))^\prime, \quad (1.10)\]
we introduce a function \(F_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s; z)\) given by
\[F_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \frac{1}{z^p(1 - z)^{-\lambda + p}} \quad (\lambda > -p). \quad (1.11)\]

Corresponding to the function \(F_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s; z)\), we introduce the linear operator
\[J_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s) : \mathcal{M}_p \to \mathcal{M}_p \]
which is defined the following convolution
\[J_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = F_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s; z) * f(z). \quad (1.12)\]

For \(f \in \mathcal{M}_p\) given by (1.1), then from (1.12), we have
\[J_{\mu,p}^\lambda(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(\lambda + p)_k (b_1)_k \cdots (b_s)_k}{(1 + \mu k)_k (a_1)_k \cdots (a_q)_k} a_k z^{k-p} \quad (1.13)\]
For convenience, we write

\[
J_{\mu,p,q,s}^\lambda (a_1) = J_{\mu,p}^\lambda (a_1, \ldots, a_q; b_1, \ldots, b_s).
\]  

(1.14)

It is easily verified from (1.13) that

\[
z (J_{\mu,p,q,s}^\lambda (a_1 + 1)f(z))' = a_1J_{\mu,p,q,s}^\lambda (a_1)f(z) - (a_1 + p)J_{\mu,p,q,s}^\lambda (a_1 + 1)f(z),
\]

(1.15)

and

\[
z (J_{\mu,p,q,s}^\lambda (a_1)f(z))' = (\lambda + p)J_{\mu,p,q,s}^\lambda (a_1)f(z) - (\lambda + 2p)J_{\mu,p,q,s}^\lambda (a_1)f(z).
\]

(1.16)

We note that

(i) for \( p = 1, \mu = 0, \lambda = \sigma - 1(\sigma > 0) \), we have \( J_{0,1,q,s}^{\sigma - 1} (a_1) = H_{\sigma,q,s} (a_1) \), where \( H_{\sigma,q,s} (a_1) \) was defined by Cho and Kim \[2\];

(ii) for \( p = s = 1, q = 2, a_1 = n + 1(n > 1), a_2 = b_1 = 0, \lambda = \sigma - 1(\sigma > 0) \), we have \( J_{0,1,1}^{\sigma - 1} (n + 1) = I_{n,\sigma} \), where \( I_{n,\sigma} \) was defined by Yuan et al. \[7\].

Next, by using the operator \( J_{\mu,p,q,s}^\lambda (a_1) \), we introduce the following classes of analytic functions for \( \phi, \psi \in P \) and \( 0 \leq \alpha, \beta < p \)

\[
\mathcal{MS}_{\mu,p,q,s}^\lambda (a_1; \alpha; \phi) = \{ f \in \mathcal{M}_p : J_{\mu,p,q,s}^\lambda (a_1) f \in \mathcal{MS}_p (\alpha; \phi) \},
\]

(1.17)

\[
\mathcal{MC}_{\mu,p,q,s}^\lambda (a_1; \alpha; \phi) = \{ f \in \mathcal{M}_p : J_{\mu,p,q,s}^\lambda (a_1) f \in \mathcal{MC}_p (\alpha; \phi) \},
\]

(1.18)

\[
\mathcal{MK}_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi) = \{ f \in \mathcal{M}_p : J_{\mu,p,q,s}^\lambda (a_1) f \in \mathcal{MK}_p (\alpha, \beta; \phi, \psi) \},
\]

(1.19)

\[
\mathcal{MQ}_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi) = \{ f \in \mathcal{M}_p : J_{\mu,p,q,s}^\lambda (a_1) f \in \mathcal{MQ}_p (\alpha, \beta; \phi, \psi) \}.
\]

(1.20)

We also note that

\[
f \in \mathcal{MC}_{\mu,p,q,s}^\lambda (a_1; \alpha; \phi) \Leftrightarrow -\frac{z f'}{p} \in \mathcal{MS}_{\mu,p,q,s}^\lambda (a_1; \alpha; \phi)
\]

(1.21)

and

\[
f \in \mathcal{MQ}_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi) \Leftrightarrow -\frac{z f'}{p} \in \mathcal{MK}_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi).
\]

(1.22)

In particular, we set

\[
\mathcal{MS}_{\mu,p,q,s}^\lambda (a_1; \frac{1 + A z}{1 + B z}) = \mathcal{MS}_{\mu,p,q,s}^\lambda (a_1; A, B) \quad (−1 < B < A \leq 1)
\]

(1.23)
and
\[ \mathcal{MC}_{\mu,p,q,s}^\lambda(a_1;\alpha;1 + A \frac{z}{1 + B}) = \mathcal{MC}_{\mu,p,q,s}^\lambda(a_1;\alpha;A,B) \quad (-1 < B < A \leq 1). \quad (1.24) \]

In this paper, we investigate several inclusion properties of the classes
\[ \mathcal{MS}_{\mu,p,q,s}^\lambda(a_1;\alpha;\phi), \mathcal{MC}_{\mu,p,q,s}^\lambda(\mu,a_1;\alpha;\phi), \mathcal{MK}_{\mu,p,q,s}^\lambda(a_1;\alpha,\beta;\phi,\psi) \text{ and } \mathcal{MQ}_{\mu,p,q,s}^\lambda(a_1;\alpha,\beta;\phi,\psi) \]
associated with the operator \( J_{\mu,p,q,s}^\lambda(a_1) \). Some applications involving integral operators are also considered.

2 Inclusion properties involving the operator \( J_{\mu,p,q,s}^\lambda(a_1) \)

The following lemmas will be required in our investigation.

**Lemma 1** [3]. Let \( \phi \) be convex univalent in \( U \) with \( \phi(0) = 1 \) and \( \Re \{\eta \phi(z) + \gamma\} > 0 \) (\( \eta,\gamma \in \mathbb{C} \)). If \( q \) is analytic in \( U \) with \( q(0) = 1 \), then
\[ q(z) + \frac{z q'(z)}{\eta q(z) + \gamma} \prec \phi(z) \quad (2.1) \]
implies \( q(z) \prec \phi(z) \).

**Lemma 2** [5]. Let \( \phi \) be convex univalent in \( U \) and let \( w \) be analytic in \( U \) with \( \Re \{w(z)\} \geq 0 \). If \( q \) is analytic in \( U \) and \( q(0) = \phi(0) \), then
\[ q(z) + w(z) z q'(z) \prec \phi(z) \quad (2.2) \]
implies \( q(z) \prec \phi(z) \).

**Theorem 1.** Let \( \phi \in P \) with
\[ \max_{z \in U} \{\Re \{\phi(z)\}\} < \min \left( \frac{\lambda + 2p - \alpha}{p - \alpha}, \frac{a_1 + p - \alpha}{p - \alpha} \right) \quad (\mu > 0; 0 \leq \alpha < p). \quad (2.3) \]

Then,
\[ \mathcal{MS}_{\mu,p,q,s}^{\lambda+1}(a_1;\alpha;\phi) \subset \mathcal{MS}_{\mu,p,q,s}^\lambda(a_1;\alpha;\phi) \subset \mathcal{MS}_{\mu,p,q,s}^\lambda(a_1 + 1;\alpha;\phi). \quad (2.4) \]

**Proof.** We begin by showing the first inclusion relationship
\[ \mathcal{MS}_{\mu,p,q,s}^{\lambda+1}(a_1;\alpha;\phi) \subset \mathcal{MS}_{\mu,p,q,s}^\lambda(a_1;\alpha;\phi). \quad (2.5) \]
Let $f \in \mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi)$ and set
\[
q(z) = -\frac{1}{p-\alpha}\left(\frac{z(J^\lambda_{\mu,p,q,s}(a_1)f(z))'}{J^\lambda_{\mu,p,q,s}(a_1)f(z)} + \alpha\right),
\]
where the function $q$ is analytic in $U$ with $q(0) = 1$. Using (1.16) and (2.6), we have
\[
-\frac{1}{p-\alpha}\left(\frac{z(J^\lambda_{\mu,p,q,s}(a_1)f(z))'}{J^\lambda_{\mu,p,q,s}(a_1)f(z)} + \alpha\right) = q(z) + \frac{zq'(z)}{\lambda + 2p - \alpha - (p-\alpha)q(z)} (z \in U). 
\]
From (2.3), we see that
\[
\Re\{\lambda + 2p - \alpha - (p-\alpha)\phi(z)\} > 0 \quad (z \in U).
\]
Applying Lemma 1 to (2.7), it follows that $q \prec \phi$, that is, $f \in \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi)$.

To prove the second part, let $f \in \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi)$ and put
\[
s(z) = -\frac{1}{p - \alpha}\left(\frac{z(J^\lambda_{\mu,p,q,s}(a_1+1)f(z))'}{J^\lambda_{\mu,p,q,s}(a_1+1)f(z)} + \alpha\right) \quad (z \in U),
\]
where the function $s$ is an analytic function with $s(0) = 1$. Then, by using the arguments similar to those detailed above with (1.15), it follows that $s \prec \phi$ in $U$, which implies that $f \in \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1+1;\alpha;\phi)$. Therefore, we complete the proof of Theorem 1.

Theorem 2. Let $\phi \in P$ with (2.3) holds. Then,
\[
\mathcal{MC}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) \subset \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi) \subset \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1+1;\alpha;\phi).
\]
Proof. Applying (1.24) and Theorem 1, we observe that
\[
f \in \mathcal{MC}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) \iff -\frac{zf'}{p} \in \mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1;\alpha;\phi) \quad \text{(from (1.21))}
\]
\[
\iff -\frac{zf'}{p} \in \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi) \quad \text{(by Theorem 1)}
\]
\[
\iff f \in \mathcal{MC}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi),
\]
and
\[
f \in \mathcal{MC}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi) \iff -\frac{zf'}{p} \in \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1;\alpha;\phi) \quad \text{(from (1.21))}
\]
\[
\iff -\frac{zf'}{p} \in \mathcal{MS}^\lambda_{\mu,p,q,s}(a_1+1;\alpha;\phi) \quad \text{(by Theorem 1)}
\]
\[
\iff f \in \mathcal{MC}^\lambda_{\mu,p,q,s}(a_1+1;\alpha;\phi),
\]
which evidently proves Theorem 2. \qed
Taking \( \phi(z) = \frac{1 + A z}{1 + B z} \) \((-1 < B < A \leq 1; z \in U\)) in Theorems 1 and 2, respectively, we have the following corollary.

**Corollary 1.** Let \( \phi \in P \) with
\[
\frac{1 + A}{1 + B} < \min\left(\frac{\lambda + 2p - \alpha}{p - \alpha}, \frac{a_1 + p - \alpha}{p - \alpha}\right) \quad (\mu > 0; 0 \leq \alpha < p; -1 < B < A \leq 1).
\]

Then,
\[
\mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha; A, B) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1; \alpha; A, B) \subset \mathcal{MS}^{\lambda}_{\mu,p,q,s}(a_1 + 1; \alpha; A, B),
\]
and
\[
\mathcal{MC}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha; A, B) \subset \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1; \alpha; A, B) \subset \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1 + 1; \alpha; A, B).
\]

Next, by using Lemma 2, we obtain the following inclusion relation for \( \mathcal{MC}^{\lambda}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi) \).

**Theorem 3.** Let \( \phi, \psi \in P \) with (2.3) holds. Then,
\[
\mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1 + 1; \alpha, \beta; \phi, \psi).
\]

**Proof.** We begin by proving that
\[
\mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi) \subset \mathcal{MK}^{\lambda}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi).
\]

Let \( f \in \mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi) \). Then, from the definition of \( \mathcal{MK}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha, \beta; \phi, \psi) \), there exists a function \( r \in \mathcal{MS}_{p}(\alpha; \phi) \) such that
\[
- \frac{1}{p - \beta} \left( \frac{z \left( J^{\lambda+1}_{\mu,p,q,s}(a_1) f(z) \right)'}{r(z)} + \beta \right) < \psi(z) \quad (z \in U).
\]

Choose the function \( g \) such that \( J^{\lambda+1}_{\mu,p,q,s}(a_1) g(z) = r(z) \). Then,
\[
g \in \mathcal{MS}^{\lambda+1}_{\mu,p,q,s}(a_1; \alpha; \phi) \quad \text{and}
\]
\[
- \frac{1}{p - \beta} \left( \frac{z \left( J^{\lambda+1}_{\mu,p,q,s}(a_1) f(z) \right)'}{J^{\lambda+1}_{\mu,p,q,s}(a_1) g(z)} + \beta \right) < \psi(z) \quad (z \in U).
\]
Now let
\[ q(z) = -\frac{1}{p - \beta} \left( \frac{z (J_{\mu,p,q,s}^\lambda (a_1) f(z))'}{J_{\mu,p,q,s}^\lambda (a_1) g(z)} + \beta \right) \quad (z \in U), \] (2.16)
where the function \( q \) is analytic in \( U \) with \( q(0) = 1 \). Using (1.16), we have
\[ (p - \beta) z q'(z) J_{\mu,p,q,s}^\lambda (a_1) g(z) + [(p - \beta) q(z) + \beta] z (J_{\mu,p,q,s}^\lambda (a_1) g(z))' = -(\lambda + p) z (J_{\mu,p,q,s}^\lambda (a_1) f(z))' + (\lambda + 2p) z (J_{\mu,p,q,s}^\lambda (a_1) f(z))'. \] (2.17)
Since \( g \in \mathcal{MS}_{\mu,p,q,s}^{\lambda + 1} (a_1; \alpha; \phi) \), by Theorem 1, we know that \( g \in \mathcal{MS}_{\mu,p,q,s}^{\lambda} (a_1; \alpha; \phi) \). Let
\[ t(z) = -\frac{1}{p - \alpha} \left( \frac{z (J_{\mu,p,q,s}^\lambda (a_1) g(z))'}{J_{\mu,p,q,s}^\lambda (a_1) g(z)} + \alpha \right). \] (2.18)
Then, using (1.16) once again, we have
\[ (\lambda + p) J_{\mu,p,q,s}^{\lambda + 1} (a_1) g(z) = \lambda + 2p - \alpha - (p - \alpha) t(z). \] (2.19)
From (2.17) and (2.19), we obtain
\[ -\frac{1}{p - \beta} \left( \frac{z (J_{\mu,p,q,s}^{\lambda + 1} (a_1)f(z))'}{J_{\mu,p,q,s}^{\lambda + 1} (a_1)g(z)} + \beta \right) = q(z) + \frac{zq'(z)}{\lambda + 2p - \alpha - (p - \alpha) t(z)} < \psi(z). \] (2.20)
Since \( \lambda > -p \) and \( t < \phi \) in \( U \) with (2.3) holds, we obtain
\[ \mathfrak{R} \{ \lambda + 2p - \alpha - (p - \alpha) t(z) \} > 0 \quad (z \in U). \]
Hence, by taking
\[ w(z) = \frac{1}{\lambda + 2p - \alpha - (p - \alpha) t(z)} \]
in equation (2.20), and then applying Lemma 2, we can show that \( q < \psi \), so that \( f \in \mathcal{MK}_{\mu,p,q,s}^{\lambda} (a_1; \alpha, \beta; \phi, \psi) \). For the second part, by using the arguments similar to those detailed above with (1.15), we obtain
\[ \mathcal{MK}_{\mu,p,q,s}^{\lambda + 1} (a_1; \alpha, \beta; \phi, \psi) \subset \mathcal{MK}_{\mu,p,q,s}^{\lambda} (a_1 + 1; \alpha, \beta; \phi, \psi). \]
Therefore, we complete the proof of Theorem 3. \( \square \)

**Theorem 4.** Let \( \phi, \psi \in P \) with (2.3) holds. Then,
\[ \mathcal{MK}_{\mu,p,q,s}^{\lambda + 1} (a_1; \alpha, \beta; \phi, \psi) \subset \mathcal{MK}_{\mu,p,q,s}^{\lambda} (a_1; \alpha, \beta; \phi, \psi) \subset \mathcal{MK}_{\mu,p,q,s}^{\lambda} (a_1 + 1; \alpha, \beta; \phi, \psi). \]
Proof. Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (1.21), we can also prove Theorem 4 by using Theorem 3 the equivalence (1.22).

3 Inclusion properties involving the integral operator \( F_{p,c}(f) \)

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator \( F_{p,c}(f) \) (see [4]) defined by

\[
F_{p,c}(f)(z) = \frac{c}{z^{c+1}} \int_0^z t^{c+p-1} f(t) \, dt \quad (f \in \mathcal{M}_c; c > 0),
\]

which satisfies the following relationship:

\[
z(J^\lambda_{\mu,p,q,s}(a_1)F_{p,c}(f)(z))' = cJ^\lambda_{\mu,p,q,s}(a_1)f(zt) - (c+p)J^\lambda_{\mu,p,q,s}(a_1)F_{p,c}(f)(z). \tag{3.1}
\]

We first prove the following inclusion relationship for the integral operator \( F_{p,c}(f) \).

**Theorem 5.** Let \( \phi \in P \) with

\[
\max_{z \in U} (\Re \{ \phi(z) \}) < \frac{c + p - \alpha}{p - \alpha} \quad (c > -p; 0 \leq \alpha < p). \tag{3.2}
\]

If \( f \in \mathcal{MS}_{\mu,p,q,s}(a_1; \alpha; \phi) \), then \( F_{p,c}(f) \in \mathcal{MS}_{\mu,p,q,s}(a_1; \alpha; \phi) \).

**Proof.** Let \( f \in \mathcal{MS}_{\mu,p,q,s}(a_1; \alpha; \phi) \) and set

\[
q(z) = -\frac{1}{p - \alpha} \left( \frac{z(J^\lambda_{\mu,p,q,s}(a_1)F_{p,c}(f)(z))'}{J^\lambda_{\mu,p,q,s}(a_1)F_{p,c}(f)(z)} + \alpha \right) \quad (z \in U), \tag{3.3}
\]

where the function \( q \) is analytic in \( U \) with \( q(0) = 1 \). From the identity (3.1), we obtain

\[
cJ^\lambda_{\mu,p,q,s}(a_1)f(z) = c + p - \alpha - (p - \alpha)q(z). \tag{3.4}
\]

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by \( z \), we have

\[
-\frac{1}{p - \alpha} \left( \frac{z(J^\lambda_{\mu,p,q,s}(a_1)f(z))'}{J^\lambda_{\mu,p,q,s}(a_1)f(z)} + \alpha \right) = q(z) + \frac{z'(z)}{(c + p - \alpha - (p - \alpha)q(z))} \cdot \phi(z) \quad (z \in U). \tag{3.5}
\]
Hence, by virtue of Lemma 1, we conclude that \( q \prec \phi \) in \( U \), which implies that
\[
F_{p,c}(f) \in \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi).
\]

**Theorem 6.** Let \( \phi \in P \) with (3.2) holds. If \( f \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi) \), then \( F_{p,c}(f) \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi) \).

**Proof.** Applying Theorem 5, it follows that
\[
f \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi) \iff -\frac{zf'}{p} \in \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi)
\]
\[
\Rightarrow F_{p,c}\left( -\frac{zf'}{p} \right) \in \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi) \quad \text{(by Theorem 5)}
\]
\[
\iff F_{p,c}(f) \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi),
\]
which proves Theorem 6.

From Theorems 5 and 6, respectively, we have the following corollary.

**Corollary 2.** Let \( \phi \in P \) with
\[
\frac{1 + A}{1 + B} < \frac{c + p - \alpha}{1 - p - \alpha} \quad (c > 0; 0 \leq \alpha < p; -1 < B < A \leq 1).
\]
Then, for the function classes defined by (1.23) and (1.24), the following inclusion relationships hold true
\[
f \in \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;A,B) \Rightarrow F_{p,c}(f) \in \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;A,B),
\]
and
\[
f \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;A,B) \Rightarrow F_{p,c}(f) \in \mathcal{MC}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;A,B).
\]

**Theorem 7.** Let \( \phi, \psi \in P \) with (3.2) holds. If \( f \in \mathcal{MK}_{\mu,p,q,s}^{\lambda}(a_1;\alpha,\beta;\phi,\psi) \), then \( F_{p,c}(f) \in \mathcal{MK}_{\mu,p,q,s}^{\lambda}(a_1;\alpha,\beta;\phi,\psi) \).

**Proof.** Let \( f \in \mathcal{MK}_{\mu,p,q,s}^{\lambda}(a_1;\alpha,\beta;\phi,\psi) \). Then, in view of the definition of \( \mathcal{MK}_{\mu,p,q,s}^{\lambda}(a_1;\alpha,\beta;\phi,\psi) \), there exists a function \( g \in \mathcal{MS}_{\mu,p,q,s}^{\lambda}(a_1;\alpha;\phi) \) such that
\[
-\frac{1}{p - \beta} \left( z \left( \frac{f_{\mu,p,q,s}^{\lambda}(a_1)}{J_{\mu,p,q,s}^{\lambda}(a_1)} \right)' \right) + \beta < \psi(z).
\]

(3.6)
Thus, we set
\[ q(z) = -\frac{1}{p - \beta} \left( z \frac{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (f) (z)}{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)} + \beta \right) \quad (z \in U), \] (3.7)
where the function \( q \) is analytic in \( U \) with \( q(0) = 1 \). Using (3.1) in (3.7), we have
\[ (p - \beta) \frac{z q' (z) J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z) + (p - \beta) q (z) J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)}{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)} = (c + p) z \frac{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (f) (z) - c J_{\mu,p,q,s}^\lambda (a_1) f (z)}{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)} \] (3.8)
Since \( g \in MS_{\mu,p,q,s}^\lambda (a_1; \alpha; \phi) \), by using Theorem 5, we obtain that \( F_{p,c} (g) \in MS_{\mu,p,q,s}^\lambda (a_1; \alpha; \phi) \). Let
\[ t(z) = -\frac{1}{p - \alpha} \left( z \frac{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)}{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)} + \alpha \right) \quad (z \in U). \] (3.9)
Then, using (3.1) once again, we have
\[ \frac{z J_{\mu,p,q,s}^\lambda (a_1) g (z)}{J_{\mu,p,q,s}^\lambda (a_1) F_{p,c} (g) (z)} = c + p - \alpha - (p - \alpha) t(z). \] (3.10)
From (3.8) and (3.10) Hence, we have
\[ -\frac{1}{p - \beta} \frac{z J_{\mu,p,q,s}^\lambda (a_1) f(z)}{J_{\mu,p,q,s}^\lambda (a_1) g(z)} + \beta = q(z) + \frac{z q'(z)}{c + p - \alpha - (p - \alpha) t(z)} \prec \psi(z). \] (3.11)
Since \( c > 0 \) and \( t \prec \phi \) in \( U \) with (3.2) holds, we have
\[ \Re \{c + p - \alpha - (p - \alpha) t(z)\} > 0. \] Hence, by taking
\[ w(z) = \frac{1}{c + p - \alpha - (p - \alpha) t(z)} \]
in (3.11) and then applying Lemma 2, we find that \( q \prec \psi \) in \( U \), so that have \( F_{p,c} (f) \in MK_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi) \). The proof of Theorem 7 is evidently completed.

**Theorem 8.** Let \( \phi, \psi \in P \) with (3.2) holds. If \( f \in MQ_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi) \), then \( F_{p,c} (f) \in MQ_{\mu,p,q,s}^\lambda (a_1; \alpha, \beta; \phi, \psi) \).
Proof. Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7.

Remark. Putting $p = 1, \mu = 0, \lambda = \sigma - 1 (\sigma > 0)$ in the above results, we obtain the results of Cho and Kim [2].

References


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