Almost Quasi-mininjective Modules

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Abstract: Let $M$ be a right $R$-module, $S = \text{End}_R(M)$. The module $M$ is called almost quasi-mininjective (or $AQ$-mininjective) if, for any simple $M$-cyclic submodule $s(M)$ of $M$, there exists a left ideal $X_s$ of $S$ such that $l_S(\text{Ker}(s)) = Ss \oplus X_s$ as left $S$-modules. In this paper, we give some characterizations and properties of $AQ$-mininjective modules.

Keywords: Almost Quasi-mininjective Modules, Endomorphism Rings

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1 Introduction

Let $R$ be a ring. A right $R$-module $M$ is called principally injective (or $P$-injective) if, every $R$-homomorphism from a principal right ideal of $R$ to $M$ can be extended to an $R$-homomorphism from $R$ to $M$. Equivalently, $l_M(r_R(a)) = Ma$ for all $a \in R$, where $l_M$ and $r_R$ are the left and right annihilators in $M$ and $R$, respectively. In [5], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. They also continued to study rings with some other kind of injectivity, namely, mininjective rings [6]. A right $R$-module $M$ is called mininjective if, every $R$-homomorphism from a simple right ideal of $R$ to $M$ can be extended to an $R$-homomorphism from $R$ to $M$, or equivalently, if $kR$ is simple, $k \in R$, $l_M(r_R(k)) = Mk$. If the regular right $R$-module $R_R$ is mininjective, then the ring $R$ is said to be a right mininjective ring. In [10], right mininjective rings are generalized to almost mininjective rings, that is, a
right $R$-module $M$ is called \textit{almost mininjective} (or $A$-mininjective) if, for any simple right ideal $kR$ of $R$, there exists an $S$-submodule $X_k$ of $M$ such that $l_M(r_R(k)) = Mk \bigoplus X_k$ as left $S$-modules. If $R_R$ is an almost mininjective module, then we call $R$ is a \textit{right almost mininjective ring}. The nice structure of almost mininjective rings draws our attention to define AQ-mininjective modules, and to investigate their characterizations and properties.

Throughout this paper, $R$ will be an associative ring with identity and all modules are unitary right $R$-modules. For right $R$-modules $M$ and $N$, $\text{Hom}_R(M,N)$ denotes the set of all $R$-homomorphisms from $M$ to $N$ and $S = \text{End}_R(M)$. A submodule $N$ of $M$ is said to be an $M$-cyclic submodule of $M$ if it is the image of an element of $S$. By notation $N \subset^c M$ ($N \subset^e M$) we mean that $N$ is a direct summand (an essential submodule) of $M$. We denote the socle and the singular submodule of $M$ by $\text{Soc}(M)$ and $Z(M)$, respectively, and that $J(M)$ denotes the Jacobson radical of $M$.

Following [7], for an $R$-module $N$ and a submodule $P$ of $N$, we will identify $\text{Hom}_R(N, M)$ with the set of maps in $\text{Hom}_R(P, M)$ that can be extended to $N$, and hence $\text{Hom}_R(N, M)$ becomes a left $S$-submodule of $\text{Hom}_R(P, M)$. In particular, for an element $s \in S$, $S$ will be regarded as a left $S$-submodule of $\text{Hom}_R(s(M), M)$.

## 2 AQ-mininjective Modules

\textbf{Definition 2.1.} Let $M$ be a right $R$-module, $S = \text{End}_R(M)$. The module $M$ is called \textit{almost quasi-mininjective} (or AQ-mininjective) if, for any simple $M$-cyclic submodule $s(M)$ of $M$, there exists a left ideal $X_s$ of $S$ such that $l_S(\text{Ker}(s)) = Ss \bigoplus X_s$ as left $S$-modules.

\textbf{Lemma 2.2.} Let $M$ be a right $R$-module and let $s(M)$ be an $M$-cyclic submodule of $M$.

1. If $\text{Hom}_R(s(M), M) = S \bigoplus Y$ as left $S$-modules, then $l_S(\text{Ker}(s)) = Ss \bigoplus X$ as left $S$-modules, where $X = \{ f_s : f \in Y \}$.

2. If $l_S(\text{Ker}(s)) = Ss \bigoplus X$ for some $X \subset S$ as left $S$-modules, then we have $\text{Hom}_R(s(M), M) = S \bigoplus Y$ as left $S$-modules, where $Y = \{ f \in \text{Hom}_R(s(M), M) : f_s \in X \}$. 
(3) \( Ss \) is a direct summand of \( l_S(Ker(s)) \) as left \( S \)-modules if and only if \( S \) is a direct summand of \( \text{Hom}_R(s(M), M) \) as left \( S \)-modules.

Proof. Define \( \theta : \text{Hom}_R(s(M), M) \to l_S(Ker(s)) \) by \( \theta(f) = f s \) for every \( f \in \text{Hom}_R(s(M), M) \). It is obvious that \( \theta \) is an \( S \)-monomorphism. For \( t \in l_S(Ker(s)) \), define \( g : s(M) \to M \) by \( g(s(m)) = t(m) \) for every \( m \in M \). Since \( Ker(s) \subset Ker(t) \), \( g \) is well-defined, so it is clear that \( g \) is an \( R \)-homomorphism. Then \( \theta(g) = gs = t \). Therefore \( \theta \) is an \( S \)-isomorphism. Let \( fs \in Ss \). Since \( fs \in l_S(Ker(s)) \), there exists \( \varphi \in Hom_R(s(M), M) \) such that \( \theta(\varphi) = fs \), so \( \varphi s = fs \).

Define \( \hat{\varphi} : M \to M \) by \( \hat{\varphi}(m) = f(m) \) for every \( m \in M \). It is clear that \( \hat{\varphi} \) is an \( R \)-homomorphism and is an extension of \( \varphi \). Then \( fs = \hat{\varphi}s = \theta(\hat{\varphi}) \). This shows that \( Ss \subset \theta(S) \). The other inclusion is clear. Then \( \theta(S) = Ss \) and \( X = \theta(Y) = \{fs : f \in Y\} \). Then the lemma follows.

From Lemma 2.2, the following corollary follows.

Corollary 2.3. Let \( M \) be a right \( R \)-module and let \( s(M) \) be an \( M \)-cyclic submodule of \( M \). Then \( l_S(Ker(s)) = Ss \) if and only if every \( R \)-homomorphism from \( s(M) \) to \( M \) can be extended to \( M \).

Theorem 2.4. The following conditions are equivalent:

1. \( M \) is \( AQ \)-mininjective.

2. There exists an indexed set \( \{X_s : s \in S\} \) of left ideals of \( S \) with the property that if \( s(M) \) is simple, \( s \in S \), then \( l_S(Im(t) \cap Ker(s)) = (X_{st} : t) + Ss \) and \( (X_{st} : t) \cap Ss \subset l_S(t) \) for all \( t \in S \), where \( (X_{st} : t)_l = \{g \in S : gt \in X_{st}\} \) if \( st \neq 0 \) and \( X_0 = 0 \).

Proof. (1) \( \Rightarrow \) (2) Let \( s(M) \) be a simple \( M \)-cyclic submodule of \( M \). Then there exists a left ideal \( X_s \) of \( S \) such that \( l_S(Ker(s)) = Ss \bigoplus X_s \) as left \( S \)-modules. Let \( t \in S \). If \( st \neq 0 \), then for any \( g \in l_S(Im(t) \cap Ker(s)) \) we have \( Ker(st) \subset Ker(gt) \). Since \( s(M) \) is simple, \( st(M) = s(M) \). Then there exists a left ideal \( X_{st} \) of \( S \) such that \( l_S(Ker(st)) = Sst \bigoplus X_{st} \) as left \( S \)-modules. Thus \( gt \in Sst \bigoplus X_{st} \) because \( gt \in l_S(Ker(gt)) \subset l_S(Ker(st)) \). Write \( gt = f(st + x) \) where \( f \in S \) and \( x \in X_{st} \). Then \( (g - f s)t = x \in X_{st} \), so \( g - f s \in (X_{st} : t)_l \). It follows that \( g \in (X_{st} : t)_l + Ss \). This shows that \( l_S(Im(t) \cap Ker(s)) \subset (X_{st} : t)_l + Ss \). Conversely, it is clear that \( Ss \subset l_S(Im(t) \cap Ker(s)) \). Let \( y \in (X_{st} : t)_l \). Then \( yt \in X_{st} \subset l_S(Ker(st)) \). If \( t(m) \in Im(t) \cap Ker(s) \), then \( st(m) = 0 \) and so \( yt(m) = 0 \).
Hence \( y \in l_S(I \cap \ker(s)) \). This shows that \((X_{st} : t)_l \subset l_S(I \cap \ker(s))\). Therefore \( l_S(I \cap \ker(s)) = (X_{st} : t)_l + Ss \). If \( gs \in (X_{st} : t)_l \cap Ss \), then \(gst \in X_{st} \cap Sst = 0\). Hence \( gs \in l_S(t)\).

(2) \( \Rightarrow \) (1) Let \( s(M) \) be a simple \( M \)-cyclic submodule of \( M \). Then there exists a left ideal \( X_s \) of \( S \) such that \( l_S(\ker(s)) = l_S(I \cap \ker(s)) = (X_s : 1)_l + Ss \) and \((X_s : 1)_l \cap Ss \subset l_S(1) = 0\). Note that \((X_s : 1)_l = X_s\). Then (1) follows.

Note that, the ring \( R \) is right almost \( \sigma \)-mininjective if and only if \( R_R \) is \( AQ \)-mininjective. From this result and Theorem 2.4 we have

**Corollary 2.5.** [10, Theorem 3.1] The following conditions are equivalent:

(i) \( R \) is right \( \sigma \)-mininjective.

(ii) There exists an indexed set \( \{X_a : a \in R\} \) of left ideals of \( R \) with the property that if \( kR \) is simple, \( k \in R \), then \( \ell[aR \cap r(k)] = (X_{ka} : a)_l + Rk \) and \((X_{ka} : a)_l \cap Rk \subset l(a) \) for all \( a \in R \), where \((X_{ka} : a)_l = \{x \in R : xa \in X_{ka}\} \) if \( ka \neq 0 \) and \( X_0 = 0 \).

Following [6], we consider the conditions \( MC_2 \) and \( MC_3 \) for a ring \( R \).

**\( MC_2 \):** If \( kR \simeq eR \) is simple, \( e = e^2 \), then \( kR = gR \), for some \( g = g^2 \).

**\( MC_3 \):** If \( eR \neq fR \) are simple, \( e = e^2, f = f^2 \), then \( eR \oplus fR = gR \), for some \( g = g^2 \).

The next proposition shows that the conditions \((MC_2)\) and \((MC_3)\) also hold in an \( AQ-\)-mininjective module.

**Proposition 2.6.** Let \( M \) be an \( AQ-\)-mininjective module and \( S = \text{End}_R(M) \).

(1) If \( e(M) \simeq k(M) \) is simple, \( e^2 = e \in S \), then \( k(M) = g(M) \), for some \( g^2 = g \in S \).

(2) If \( e(M) \neq f(M) \) are simple, \( e^2 = e \in S, f^2 = f \in S \), then \( e(M) \oplus f(M) = g(M) \), for some \( g^2 = g \in S \).

**Proof.** (1) Let \( e(M) \simeq k(M) \) is a simple submodule of \( M \), \( e^2 = e \in S \), and let \( \sigma : e(M) \rightarrow k(M) \) be an \( R-\)isomorphism. Set \( \alpha = \sigma e \). Then \( \alpha(M) = k(M) \) and \( \ker(e) = \ker(\alpha) \), so \( \alpha(M) \) is a simple submodule of \( M \). Then \( e \in l_S(\ker(e)) \). By [6, Theorem 3.1], \( l_S(\ker(e)) = S \alpha \oplus X_{\alpha} \). Thus \( X_{\alpha} = \text{a left ideal of } S \), and so \( e = a \alpha \) for some \( \alpha \in S \). Write \( e = sa + x \) where \( s \in S \) and \( x \in X_{\alpha} \). Then \( e = sa + x \) and so \( \alpha - a \alpha = ax \in S \alpha \cap X_{\alpha} = 0 \), hence \( \alpha = a \alpha \). Put \( g = a \alpha \). Then \( g^2 = g \) and \( k(M) = g(M) \).
(2) Let \( e(M) \neq f(M) \) are simple, \( e^2 = e \in S, f^2 = f \in S \). Then we have \( e(M) \bigoplus f(M) = e(M) \bigoplus (1 - e)f(M) \). If \( (1 - e)f(M) = 0 \), then \( e(M) \bigoplus f(M) = e(M) \), because by assumption we have \( e(M) \cap f(M) = 0 \). Hence \( e(M) \bigoplus f(M) \) is a direct summand of \( M \). If \( (1 - e)f(M) \neq 0 \), then \( f(M) \simeq (1 - e)f(M) \) so \( (1 - e)f(M) = g(M) \), \( g^2 = g \in S \), by (1). Then \( eg = 0 \) so \( h = e + g - ge \) is an idempotent such that \( he = e = eh \) and \( hg = g = gh \). If \( x \in e(M) \bigoplus f(M) \), then \( x \in e(M) \bigoplus (1 - e)f(M) = e(M) \bigoplus g(M) \). Write \( x = e(m) + g(n) \). It follows that \( x = he(m) + hg(n) = h(e(m) + g(n)) \in h(M) \). This shows that \( e(M) \bigoplus f(M) \subset h(M) \). The other inclusion is clear. Then \( e(M) \bigoplus f(M) = h(M) \).

**Proposition 2.7.** Let \( M \) be an \( AQ \)-mininjective module which is a principal self-generator. Then \( Soc(M_R) \subset r_M(J(S)) \).

**Proof.** Let \( mR \) be a simple submodule of \( M \). Suppose \( \alpha(m) \neq 0 \) for some \( \alpha \in J(S) \). As \( M \) is a principal self-generator, \( mR = \sum_{s \in I} s(M) \) for some \( I \subset S \). Since \( mR \) is a simple, \( mR = s(M) \) for some \( 0 \neq s \in I \). Then \( \alpha s \neq 0 \) and \( \text{Ker}(\alpha s) = \text{Ker}(s) \). Note that \( \alpha s(M) \) is a nonzero homomorphic image of the simple module \( s(M) \), then \( \alpha s(M) \) is simple. Since \( M \) is \( AQ \)-mininjective, there exists a left ideal \( X_{\alpha s} \) of \( S \) such that \( l_S(\text{Ker}(\alpha s)) = \alpha s \bigoplus X_{\alpha s} \) as left \( S \)-modules. Thus \( l_S(\alpha s) = \alpha s \bigoplus X_{\alpha s} \). Write \( s = \beta \alpha s + x \) where \( \beta \in S \) and \( x \in X_{\alpha s} \). Then \( (1 - \beta \alpha)s = x \) and so \( s = (1 - \beta \alpha)^{-1}x \in X_{\alpha s} \). It follows that \( \alpha s \in \alpha s \bigcap X_{\alpha s} = 0 \), a contradiction.

The following corollary follows from Proposition 2.7 and [8, 21.15].

**Corollary 2.8.** Let \( M \) be an \( AQ \)-mininjective module which is a principal self-generator. If \( S \) is semilocal, then \( Soc(M_R) \subset Soc(SM) \).

Let \( M \) be a right \( R \)-module with \( S = \text{End}_R(M) \). Following [4], write \( \Delta = \{ s \in S : \text{ker}(s) \subset^e M \} \). It is known that \( \Delta \) is an ideal of \( S \) [4, Lemma 3.2].

**Proposition 2.9.** Let \( M \) be an \( AQ \)-mininjective module which is a principal self-generator and \( Soc(M_R) \subset^e M \). Then \( J(S) \subset \Delta \).

**Proof.** Let \( s \in J(S) \). If \( \text{Ker}(s) \not\subset^e M \), then \( \text{Ker}(s) \cap N = 0 \) for some nonzero submodule \( N \) of \( M \). Since \( Soc(M_R) \subset^e M \), \( Soc(M_R) \cap N \neq 0 \). Then there exists a simple submodule \( kR \) of \( M \) such that \( kR \subset Soc(M_R) \cap N \) [1, Corollary 9.10]. As \( M \) is a principal self-generator and \( kR \) is simple, \( kR = t(M) \) for some \( t \in S \). It follows that \( \text{Ker}(st) = \text{Ker}(t) \). Since \( ts(M) \) is a nonzero homomorphic image
of the simple module $t(M)$, $st(M) = t(M)$. Then there exists a left ideal $X_{st}$ of $S$ such that $t \in l_S(\ker(t)) = l_S(\ker(st)) = Sst \bigoplus X_{st}$. Write $t = \alpha st + x$ where $\alpha \in S$ and $x \in X_{st}$. It follows that $t = (1 - \alpha s)^{-1} x$. Then $st = s(1 - \alpha s)^{-1} x \in Sst \cap X_{st} = 0$, a contradiction.

**Proposition 2.10.** Let $M$ be an $AQ$–mininjective module which is a principal self-generator and $\text{Soc}(M_R) \subset e M$. If $M$ is nonsingular, then $J(S) = 0$.

**Proof.** Since $J(S) \subset \triangle$ by Proposition 2.9, we show that $\triangle = 0$. Let $s \in \triangle$ and let $m \in M$. Define $\varphi : R \to M$ by $\varphi(r) = mr$. It is clear that $\varphi$ is an $R$–homomorphism. Thus

$$r_R(s(m)) = \{ r \in R : s(mr) = 0 \}$$

$$= \{ r \in R : mr \in \ker(s) \}$$

$$= \{ r \in R : \varphi(r) \in \ker(s) \}$$

$$= \varphi^{-1}(\ker(s)).$$

It follows that $\varphi^{-1}(\ker(s)) \subset e R$ [3, Lemma 5.8(a)] so $r_R(s(m)) \subset e R$. Thus $s(m) \in Z(M_R) = 0$ because $M$ is nonsingular. As this is true for all $m \in M$, we have $s = 0$. Hence $\triangle = 0$ as required.

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